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FINITE TRANSFORM SOLUTION OF THE TEMPERATURE OF A PLATE HEATED BY A MOVING DISCRETE SOURCE†

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Abstract—The conduction equation is solved for the temperature distribution in a thin plate having convection losses on all boundaries and being heated by a moving discrete source. The solution is based on the properties of an Hermitian operator and its orthogonal basis vectors.

NOMENCLATURE

A, B,	constants of integration;
a,	width of plate [ft];
b,	thickness of plate [ft];
c_1, c_2,	constants of integration;
c_n,	constant;
F,	function;
f,	transform of F;
h,	convection coefficient [Btu/h-ft^2-degR];
i,	integer;
J,	total number of contacts;
j,	integer;
k,	conductivity [Btu/h-ft-degR];
L,	linear differential operator;
L^*,	adjoint operator;
l,	length of plate [ft];
N_{Bi},	Biot Number [dimensionless];
Q,	source [Btu/h-ft^3];
Q_j,	energy released at jth contact [Btu/ft];
q,	transform of Q;
t,	time [h];
$\mathbf{u}(x)$,	vector function of x;
$\mathbf{v}(x)$,	vector function of x;
W,	temperature [degR];
W_s,	temperature of surroundings [degR];
w,	transform of W;
x,	distance [ft];
y,	distance [ft];
z,	distance [ft].

Greek symbols

α,	thermal diffusivity [ft^2/h];
β,	eigenvalue [dimensionless];
γ,	eigenvalue [1/ft];
δ,	Dirac delta function;
θ,	eigenvalue [1/ft];
λ,	eigenvalue [1/ft^2];
σ_n,	constant in $u_n(\gamma_n x)$ [dimensionless];
$\psi_{m,n}$,	derived constant [1/h].

INTRODUCTION

THE TEMPERATURE distribution in solids due to a moving source has extensive application in sliding friction, internal ballistics, machining and in metal treating operations such as welding, casting, quenching and flame hardening [1].

Spraragen and Clausen [2], in 1937, reviewed the literature on the subject of welding. In a classic paper, Rosenthal [3], in 1946, developed the quasi-steady state theory for a uniform source moving at a uniform velocity in an infinite medium in the direction of motion. Carslaw and Jaeger [4] give solutions to problems of this type using sources and sinks, and also by the use of Green's function.

Slinn [5] in a recent article, points out the advantage of using transforms over traditional

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methods to solve linear partial differential equations, because of increased generality of the method, and the ease of solution when the transform properties are known.

In this paper, using methods discussed by Lanczos [6] concerning linear differential operators, the operator is shown to be Hermitian, its eigenstructure is analyzed, an inner product and a transform are defined, and an identity concerning the transform of the second derivative is developed. Using the transform identity, a very general type of solution for the temperature distribution in a thin plate subject to the heating of a moving discrete source is determined.

PROBLEM

The conduction equation for a thin plate with a source and losses on all the boundaries by convection (covers the range from constant temperature boundaries to insulated boundaries) is

$$\frac{\partial^2 W(x,y,t)}{\partial x^2} + \frac{\partial^2 W(x,y,t)}{\partial y^2} - \frac{(h_1 + h_2)W(x,y,t)}{kb} + \frac{Q(x,y,t)}{k} = \frac{1}{\alpha}\frac{\partial W(x,y,t)}{\partial t} \qquad (1)$$

where

W, temperature
h_1 convection coefficient, upper surface;
h_2. convection coefficient. lower surface;
k, conductivity;
b, thickness of plate;
Q, source;
α, thermal diffusivity.

The boundary and initial conditions for a plate having losses at the edges due to a convection coefficient h when the surroundings, W_s, are at zero (see Fig. 1) are:

1. $\dfrac{\partial W(0,y,t)}{\partial x} = \dfrac{hW(0,y,t)}{k}$,

2. $\dfrac{\partial W(l,y,t)}{\partial x} = \dfrac{-hW(l,y,t)}{k}$,

3. $\dfrac{\partial W(x,0,t)}{\partial y} = \dfrac{hW(x,0,t)}{k}$,

4. $\dfrac{\partial W(x,a,t)}{\partial y} = \dfrac{-hW(x,a,t)}{k}$

5. $W(x,y,0) = F(x,y)$.

To solve equation (1) subject to the boundary and initial conditions shown, it is necessary to develop the properties of a special linear operator L.

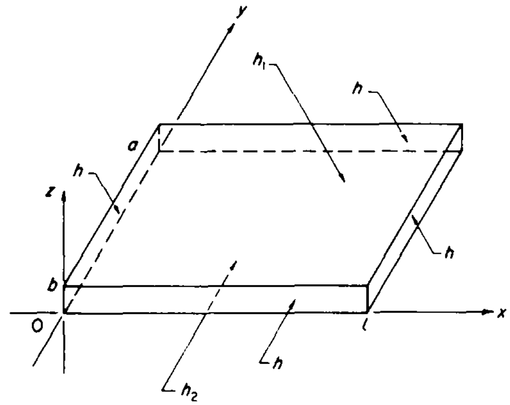

FIG. 1. Thin plate boundary conditions.

HERMITIAN CHARACTER OF L

The linear differential operator, we require, that operates on $\mathbf{u}$ is

$$L\mathbf{u} = \frac{d^2u}{dx^2} \qquad (2)$$

where the relationship between a vector $\mathbf{u}$ and its components is analogous to

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \qquad (3)$$

in the discrete case, in some basis. The operator L is defined in the domain,

$$D(L): u(x); \quad 0 < x < l$$
$$u'(0) = \frac{h}{k} u(0)$$
$$u'(l) = -\frac{h}{k} u(l)$$
$$0 < h < \infty$$
$$0 < k < \infty$$

and has the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_0^l \overline{u(x)}\, v(x)\, \mathrm{d}x \tag{4}$$

where the bar denotes complex conjugate.

Using Lanczos' bilinear identity [7], where L^* is the adjoint operator, we have

$$\langle \mathbf{v}, L\mathbf{u} \rangle = \langle L^*\mathbf{v}, \mathbf{u} \rangle \tag{5}$$

and integrating by parts gives

$$\langle L^*\mathbf{v}, \mathbf{u}, \rangle = \bar{v} \frac{\mathrm{d}u}{\mathrm{d}x}\bigg|_0^l - \frac{\mathrm{d}\bar{v}}{\mathrm{d}x} u \bigg|_0^l + \int_0^l \frac{\overline{\mathrm{d}^2 v}}{\mathrm{d}x^2} u \,\mathrm{d}x. \tag{6}$$

If the adjoint operator L^* is defined in the domain,

$$D(L^*): v(x); \quad 0 < x < l$$
$$v'(0) = \frac{h}{k} \mathrm{v}(0)$$
$$v'(\mathrm{l}) = -\frac{h}{k} \mathrm{v}(\mathrm{l})$$
$$0 < h < \infty$$
$$0 < k < \infty$$

i.e. the same domain as the operator L, then

$$\langle L^*\mathbf{v}, \mathbf{u} \rangle = \int_0^l \frac{\overline{\mathrm{d}^2 v}}{\mathrm{d}x^2} u \,\mathrm{d}x = \int_0^l \overline{L^* v}\, u \,\mathrm{d}x. \tag{7}$$

It follows then that

$$L^* v = \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \tag{8}$$

and

$$L^* = \frac{\mathrm{d}^2}{\mathrm{d}x^2} = L \tag{9}$$

and so L is an Hermitian (self-adjoint) operator. It is known that an Hermitian operator has an orthogonal basis consisting of the eigenvectors of L.

EIGENSPACE OF THE HERMITIAN OPERATOR L

To determine the eigenvectors of L, we must solve the equation

$$L\mathbf{u} = \lambda \mathbf{u} \tag{10}$$

$\lambda = 0$:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0 \tag{11}$$

$$u(x) = Ax + B. \tag{12}$$

Boundary conditions:

1. $u'(0) = \frac{h}{k} u(0)$
2. $u'(l) = -\frac{h}{k} u(l)$

Using the boundary conditions listed gives the eigenvector solution

$$\mathbf{u}(x) = 0 \tag{13}$$

and there are no eigenvectors for the eigenvalue $\lambda = 0$, and thus the null space $N(L)$ of the operator is empty.

$\lambda \neq 0$:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \lambda u \tag{14}$$

Defining

$$\lambda = -\gamma^2 \tag{15}$$

results in the solution of equation (14) as

$$u(x) = c_1 \cos \gamma x + c_2 \sin \gamma x \tag{16}$$

Using the boundary conditions gives the eigenvalue equation

$$\tan \beta = \frac{2\beta N_{Bi}}{\beta^2 - {N_{Bi}}^2} \tag{17}$$

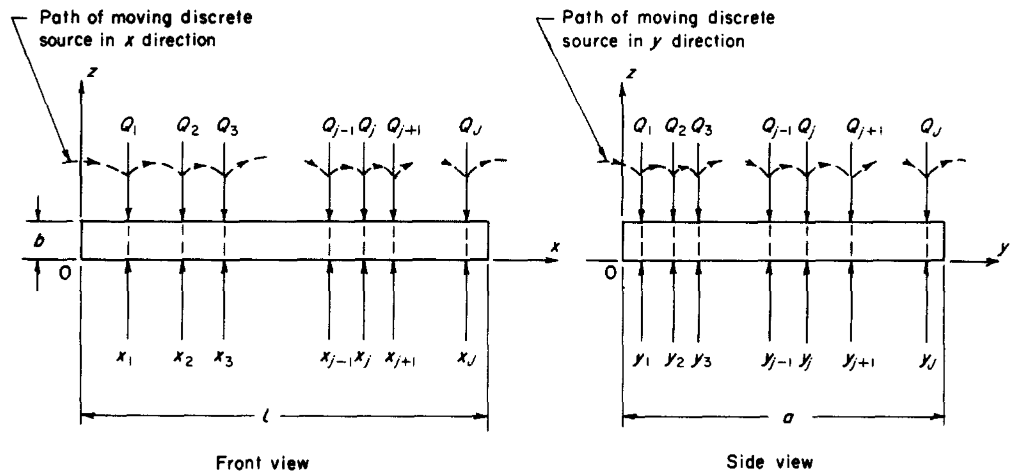

FIG. 2. Thin plate heated by a moving discrete source.

where

$$\beta = \gamma l \tag{18}$$

$$N_{Bi} = \frac{hl}{k} \tag{19}$$

and

$$\beta = \beta_1, \beta_2, \beta_3, \ldots, \beta_n, \ldots \tag{20}$$

Plots of the first ten eigenvalues for various arguments of N_{Bi} are shown in Fig. 3–9. Thus the eigenvalues of L are

$$\lambda = \lambda_n = -\gamma_n^2 = -\left(\frac{\beta_n}{l}\right)^2 \tag{21}$$

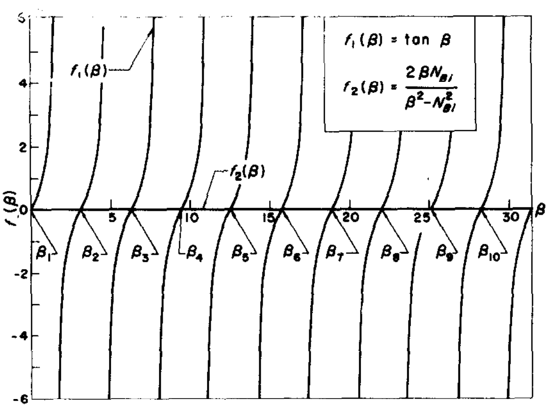

FIG. 3. Eigenvalues for $N_{Bi} = 0{\cdot}001$.

and the eigenvectors are

$$\mathbf{u}_n(\gamma_n x) = \cos\gamma_n x + \sigma_n \sin\gamma_n x \tag{22}$$

where

$$\sigma_n = \frac{h}{k\gamma_n}. \tag{23}$$

An arbitrary vector $\mathbf{F}(x)$ can be expanded in this basis by the expansion

$$\mathbf{F}(x) = \sum_{n=1}^{\infty} c_n \mathbf{u}_n(\gamma_n x) \tag{24}$$

where

$$c_n = \frac{\langle \mathbf{u}_n, \mathbf{F}\rangle}{\langle \mathbf{u}_n, \mathbf{u}_n\rangle} = \frac{2\beta_n^2 \langle \mathbf{u}_n, \mathbf{F}\rangle}{l[\beta_n^2 + N_{Bi}^2 + 2N_{Bi}]}. \tag{25}$$

TRANSFORM OF THE SECOND-ORDER DERIVATIVE

If we define the following inner product as the transform of F,

$$\langle \mathbf{u}_n, \mathbf{F}\rangle = \int_0^l \overline{u_n(\gamma_n x)}\, F(x)\, \mathrm{d}x = f(\gamma_n) \tag{26}$$

then

$$\langle \mathbf{u}_n, \mathbf{F}''\rangle = \int_0^l \overline{u_n(\gamma_n x)} \frac{\mathrm{d}^2 F}{\mathrm{d}x^2}(x)\, \mathrm{d}x. \tag{27}$$

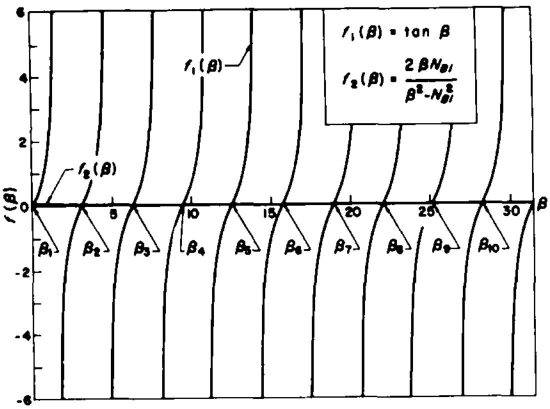

FIG. 4. Eigenvalues for $N_{Bi} = 0{\cdot}01$.

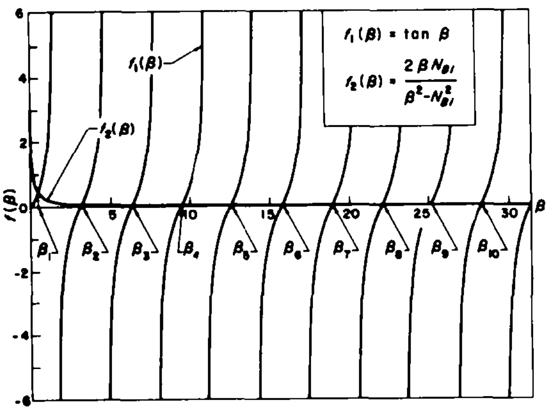

FIG. 5. Eigenvalues for $N_{Bi} = 0{\cdot}1$.

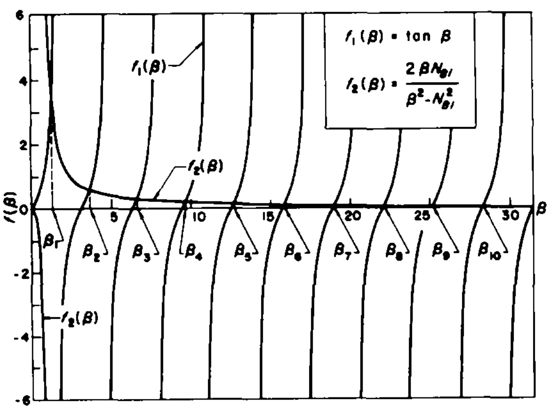

FIG. 6. Eigenvalues for $N_{Bi} = 1$.

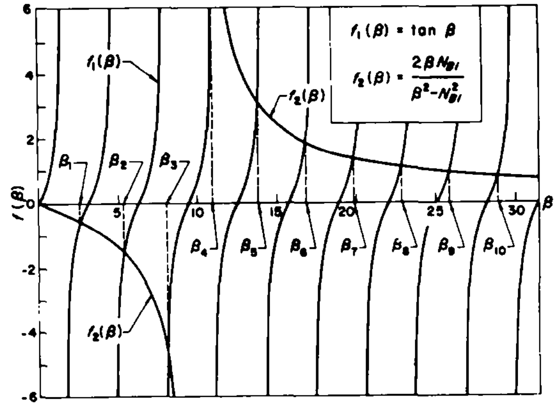

FIG. 7. Eigenvalues for $N_{Bi} = 10$.

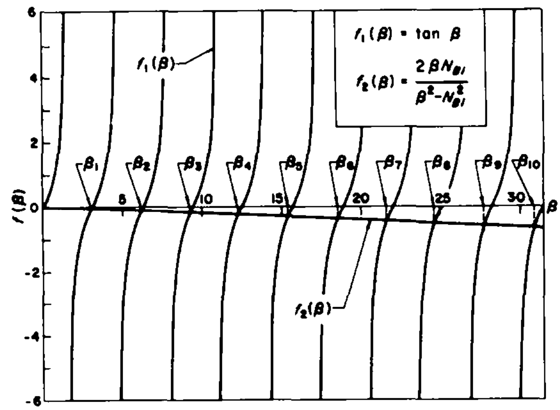

FIG. 8. Eigenvalues for $N_{Bi} = 100$.

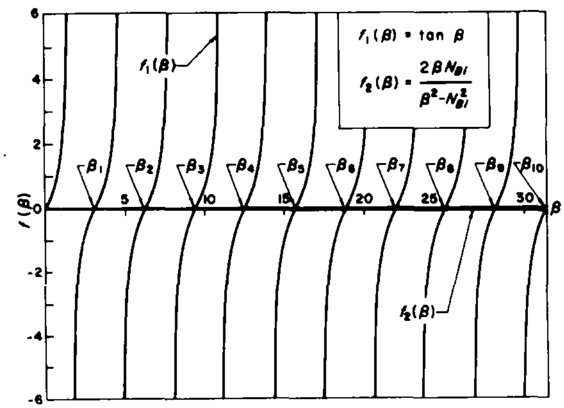

FIG. 9. Eigenvalues for $N_{Bi} = 1000$.

Integrating by parts, and using the known properties of $\mathbf{u}_n(\gamma_n x)$, we obtain the transform identity

$$\langle \mathbf{u}_n, \mathbf{F}'' \rangle = u(\beta_n)\left[F'(l) + \frac{h}{k}F(l)\right] - u_n(0)\left[F'(0) - \frac{h}{k}F(0)\right] - \gamma_n^2 f(\gamma_n). \quad (28)$$

NATURE OF MOVING DISCRETE SOURCE

For a moving source that releases energy in arbitrary, discrete amounts, at arbitrary, discrete points (in the manner of spotwelding) (see Fig. 2), the source term is given by

$$Q(x, y, t) = \sum_{j=1}^{J} Q_j\, \delta(x - x_j)\, \delta(y - y_j)\, \delta(t - t_j). \quad (29)$$

SOLUTION

Taking transforms of equation (1) and the boundary and initial conditions by means of the transform

$$\langle \mathbf{u}_m, \mathbf{F} \rangle = \int_0^l \overline{u_m(\gamma_m x)}\, F(x)\, \mathrm{d}x = \int_0^l F(x)\, u_m(\gamma_m x)\, \mathrm{d}x = f(\gamma_m) \quad (30)$$

where

$$\gamma_m l = \beta_m, \qquad m = 1, 2, 3, \ldots \quad (18a)$$

gives rise to the partial differential equation

$$\frac{\partial^2 w}{\partial y^2}(\gamma_m, y, t) - \left[\gamma_m^2 + \frac{(h_1 + h_2)}{kb}\right] w(\gamma_m, y, t) + \frac{q}{k}(\gamma_m, y, t) = \frac{1}{\alpha}\frac{\partial w}{\partial t}(\gamma_m, y, t) \quad (31)$$

and its accompanying boundary and initial conditions:

1. $\frac{\partial w}{\partial y}(\gamma_m, 0, t) = \frac{h}{k} w(\gamma_m, 0, t)$
2. $\frac{\partial w}{\partial y}(\gamma_m, a, t) = -\frac{h}{k} w(\gamma_m, a, t)$
3. $w(\gamma_m, y, 0) = f(\gamma_m, y)$

and where

$$q(\gamma_m, y, t) = \int_0^l Q(x, y, t)\, u_m(\gamma_m x)\, \mathrm{d}x \quad (32)$$

and

$$f(\gamma_m, y) = \int_0^l F(x, y)\, u_m(\gamma_m x)\, \mathrm{d}x. \quad (33)$$

Taking transforms of equation (31) and its accompanying initial condition by means of the transform

$$\langle \mathbf{u}_n, \mathbf{F} \rangle = \int_0^a \overline{u_n(\theta_n y)}\, F(y)\, \mathrm{d}y = \int_0^a F(y)\, u_n(\theta_n y)\, \mathrm{d}y = f(\theta_n) \quad (34)$$

where

$$\theta_n a = \beta_n, \qquad n = 1, 2, 3, \ldots \quad (18b)$$

gives rise to the ordinary differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}t}(\gamma_m, \theta_n, t) + \alpha\left[\gamma_m^2 + \theta_n^2 + \frac{(h_1 + h_2)}{kb}\right] w(\gamma_m, \theta_n, t) = \frac{\alpha}{k} q(\gamma_m, \theta_n, t) \tag{35}$$

and its initial condition:

1. $w(\gamma_m, \theta_n, 0) = f(\gamma_m, \theta_n)$

where

$$q(\gamma_m, \theta_n, t) = \int_0^a q(\gamma_m, y, t)\, u_n(\theta_n y)\, \mathrm{d}y \tag{36}$$

and

$$f(\gamma_m, \theta_n) = \int_0^a f(\gamma_m, y)\, u_n(\theta_n y)\, \mathrm{d}y. \tag{37}$$

The solution to equation (35) using the initial condition is

$$w(\gamma_m, \theta_n, t) = f(\gamma_m, \theta_n) \exp(-\psi_{m,n} t) + \alpha/k\, q(\gamma_m, \theta_n, t) * \exp(-\psi_{m,n} t) \tag{38}$$

where

$$\psi_{m,n} = \alpha\left[\gamma_m^2 + \theta_n^2 + \frac{(h_1 + h_2)}{kb}\right] \tag{39}$$

and

$$q(\gamma_m, \theta_n, t) * \exp(-\psi_{m,n} t) = \int_0^t q(\gamma_m, \theta_n, \tau) \exp[-\psi_{m,n}(t - \tau)]\, \mathrm{d}\tau. \tag{40}$$

Using equation (29), equation (40) becomes

$$q(\gamma_m, \theta_n, t) * \exp(-\psi_{m,n} t) = \sum_{j=1}^{J} Q_j\, u_m(\gamma_m x_j)\, u_n(\theta_n y_j) \exp[-\psi_{m,n}(t - t_j)] \tag{40a}$$

and equation (38) can be written as

$$w(\gamma_m, \theta_n, t) = f(\gamma_m, \theta_n) \exp(-\psi_{m,n} t) + \alpha/k \sum_{j=1}^{J} Q_j\, u_m(\gamma_m x_j)\, u_n(\theta_n y_j) \exp[-\psi_{m,n}(t - t_j)]. \tag{38a}$$

Using the expansion equation the solution can be written as

$$\mathbf{W}(x, y, t) = \sum_{m=1}^{\infty} c_m\, u_m(\gamma_m x) = \sum_{m=1}^{\infty} \frac{\langle \mathbf{u}_m, \mathbf{W}(x, y, t)\rangle}{\langle \mathbf{u}_m, \mathbf{u}_m\rangle} \mathbf{u}_m(\gamma_m x)$$

$$= \frac{2}{l} \sum_{m=1}^{\infty} \frac{\beta_m^2 w(\gamma_m, y, t)}{[\beta_m^2 + N_{Bi}^2(l) + 2N_{Bi}(l)]} \mathbf{u}_m(\gamma_m x) \tag{41}$$

where

$$N_{Bi}(l) = \frac{hl}{k} \tag{42}$$

and similarly

$$\mathbf{w}(\gamma_m, y, t) = \sum_{n=1}^{\infty} c_n \mathbf{u}_n(\theta_n y) = \sum_{n=1}^{\infty} \frac{\langle \mathbf{u}_n, \mathbf{w}(\gamma_m, y, t)\rangle}{\langle \mathbf{u}_n, \mathbf{u}_n\rangle} \mathbf{u}_n(\theta_n y)$$

$$= \frac{2}{a} \sum_{n=1}^{\infty} \frac{\beta_n^2 \, w(\gamma_m, \theta_n, t)}{[\beta_n^2 + N_{Bi}^2(a) + 2N_{Bi}(a)]} \mathbf{u}_n(\theta_n y) \tag{43}$$

where

$$N_{Bi}(a) = \frac{ha}{k}. \tag{44}$$

Dropping the vector notation, the solution becomes

$$W(x, y, t) = \frac{4}{al} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\beta_m^2 \, \beta_n^2 \, w(\gamma_m, \theta_n, t) \, u_m(\gamma_m x) \, u_n(\theta_n y)}{[\beta_m^2 + N_{Bi}^2(l) + 2N_{Bi}(l)][\beta_n^2 + N_{Bi}^2(a) + 2N_{Bi}(a)]} \tag{45}$$

or in a slightly expanded form

$$W(x, y, t) = \frac{4}{al} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\beta_m^2 \beta_n^2 \, u_m(\gamma_m x) \, u_n(\theta_n y)}{[\beta_n^2 + N_{Bi}^2(l) + 2N_{B_i}(l)][\beta_n^2 + N_{Bi}^2(a) + 2N_{Bi}(a)]}$$

$$\{\int_0^a \int_0^l F(x, y) \, u_m(\gamma_m x) \, u_n(\theta_n y) \, \mathrm{d}x \, \mathrm{d}y + \alpha/k \sum_{j=1}^{J} Q_j \, u_m(\gamma_m x_j) \, u_n(\theta_n y_j) \exp{(\psi_{m,n} t_j)}\} \exp{(-\psi_{m,n} t)} \tag{46}$$

for $0 < x < l, 0 < y < a$, and $t \leqslant t_J$.

For

$$t_i \leqslant t < t_{i+1}, i = 1, 2, 3, \ldots \tag{47}$$

where

$$1 \leqslant i < J \tag{48}$$

the solution is the same as equation (46), except that the summation on j stops at i instead of J.

For

$$0 \leqslant t < t_1 \tag{49}$$

then the solution, equation (46), has none of the terms involving the summation on j.

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Résumé—L'équation de la conduction est résolue pour la distribution de température dans une plaque mince ayant des pertes par convection sur toutes ses frontières et chauffée par une source ponctuelle en mouvement. La solution est basée sur les propriétés d'un opérateur hermitien et de ses vecteurs orthogonaux de base.

Zusammenfassung—Die Leitungsgleichung wird gelöst für die Temperaturverteilung in einer dünnen Platte, die von einer bewegten Einzelquelle beheizt wird und Konvektionsverluste an allen Berandungen aufweist. Die Lösung beruht auf den Eigenschaften eines Hermite Operators und orthogonalen Basisvektoren.

Аннотация—Решение уравнения теплопроводности используется для получения температурного распределения в тонкой пластине, нагреваемой движущимся дискретным источником при наличии конвективных теплопотерь на всех границах. Решение основано на свойствах оператора Эрмита и его ортогональных базисных векторах.